# Partial rigidity on $\mathcal{S}$-adic subshifts

Dyadisc 8: Approximations of Symbolic Systems
UPJV, Amiens, France

Tristán Radić

Northwestern University

Joint work with A. Maass and S. Donoso.

**Definition** $(X, \mathcal{X}, \mu, T)$ is partially rigid if there exists $\delta > 0$ and a sequence $(N_n)_{n\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} \mu(A \cap T^{-N_n}A) \geq \delta\mu(A) \quad \text{for all} \quad A \in \mathcal{X}$$

- $(N_n)_{n\in\mathbb{N}}$ is a *partial rigidity sequence*.
- $\delta$ is a *constant of partial rigidity* ($\delta > \gamma > 0$ is another constant of partial rigidity).

**Special case:** The system is **rigid** whenever $\delta = 1$, in that case

$$\lim_{n\to\infty} \mu(A \cap T^{-N_n}A) = \mu(A) \quad \text{for all} \quad A \in \mathcal{X}.$$

# Partial rigidity

**Facts:** Partially rigid systems are

- not mixing,
- zero entropy (Bruin, Karpel, Oprocha and Radinger 2025).

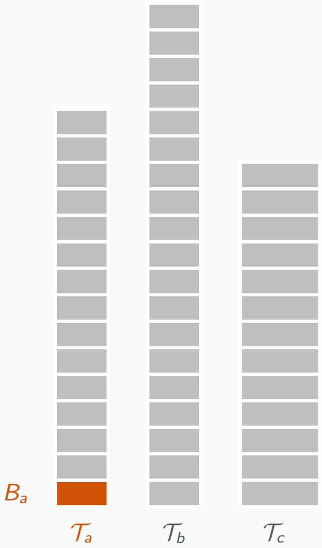
**Examples:**

- Equicontinuous systems are *rigid*
- Substitution subshifts are *partially rigid* (Dekking 1978)
- Interval exchanges transformations are *partially rigid* (Katok 1980)
- Linearly recurrent subshifts are *partially rigid* (Cortez, Durand, Host, Maass, 2003)
- Cantor sytems of finite exact rank are *partially rigid* (Bezuglyi, Kwiatkowski, Medynets and Solomyak 2013 - Danilenko 2016)
- Non-superlinear complexity subshifts are *partially rigid* (Creutz 2022) this include the rest of the cases

# How to characterize partial rigidity?

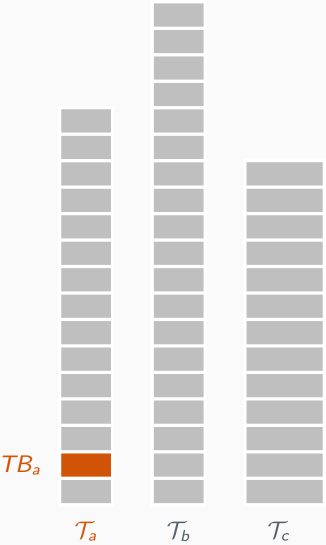
# Kakutani-Rokhlin

- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.

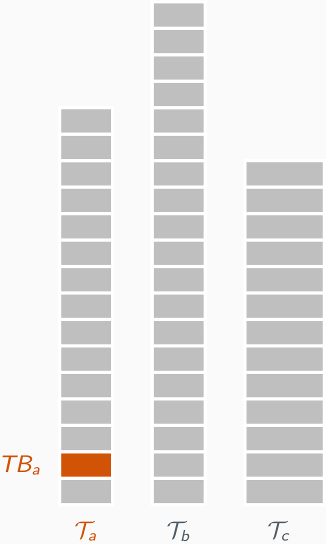


# Kakutani-Rokhlin

- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.

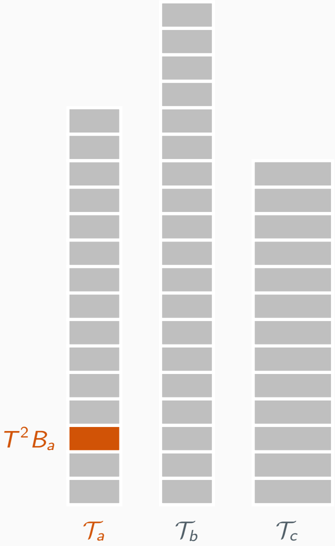


- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.



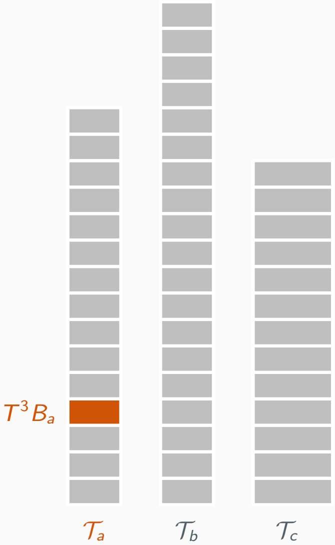
# Kakutani-Rokhlin

- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.



# Kakutani-Rokhlin

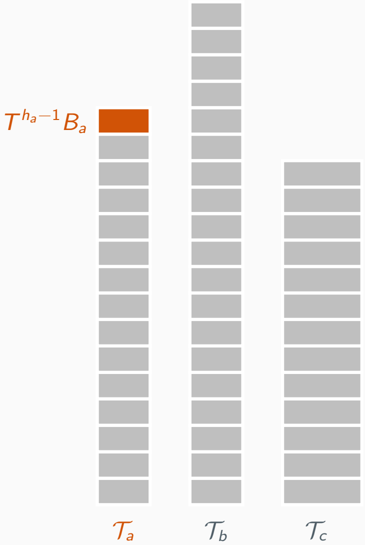
- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.



...

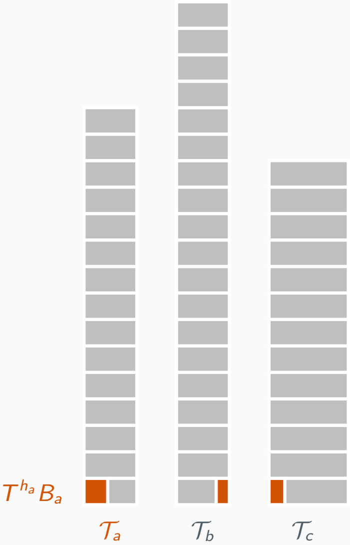
# Kakutani-Rokhlin

- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.

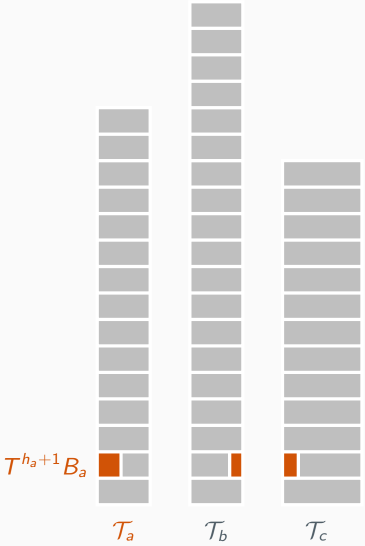


# Kakutani-Rokhlin

- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.



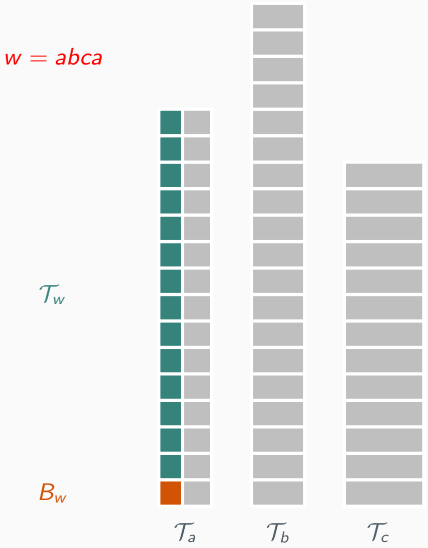
- $\mathcal{T}_i$ are **towers**
- $B_i$ are the **bases** of $\mathcal{T}_i$
- $\bigcup \mathcal{T}_i = X$.



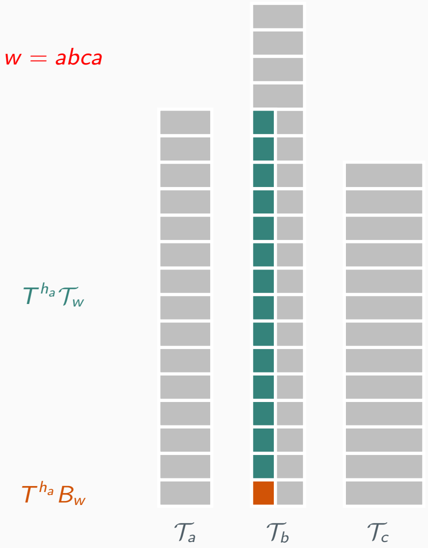
**Partial rigidity is a notion of recurrence and it is captured by the complete return words.**

- Take a finite alphabet $A$ (e.g. $A = \{a, b, c\}$)
- $w = w_1 w_2 \cdots w_\ell \in A^*$ is a complete return word if $w_1 = w_\ell$ (e.g. $w = abca$ or $w = bccbab$).

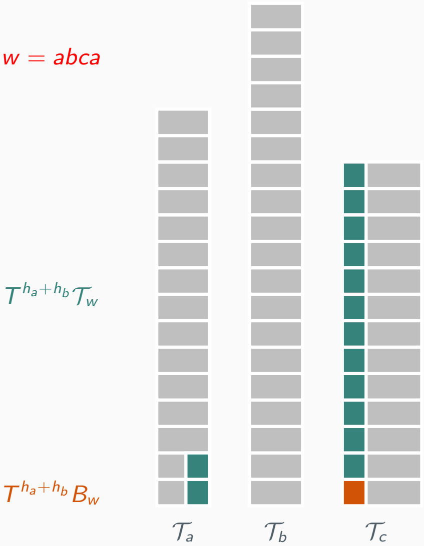
**With this words we define sub-towers.**

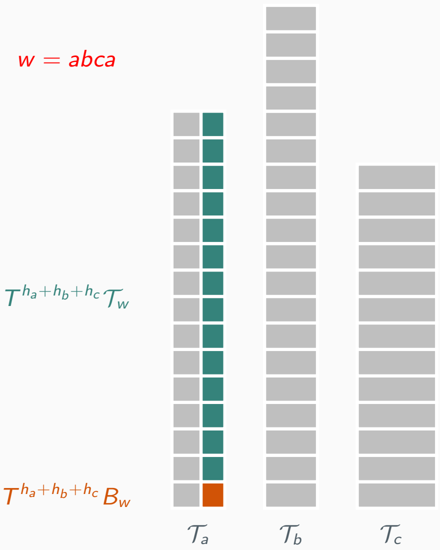

$w = abca$
$\mathcal{T}_w$
$B_w$
$\mathcal{T}_a$
$\mathcal{T}_b$
$\mathcal{T}_c$

# Kakutani-Rokhlin



# Kakutani-Rokhlin




$w = abca$
$T^{h_a+h_b+h_c}\mathcal{T}_w$
$T^{h_a+h_b+h_c}B_w$
$\mathcal{T}_a$
$\mathcal{T}_b$
$\mathcal{T}_c$

**Theorem** (Danilenko 2016) Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $(\mathcal{P}^{(n)})_{n\in\mathbb{N}}$ be a sequence of Kakutani-Roklin partition of $X$ such that

(I) $\mathcal{P}^{(n)} = \{\mathcal{T}_1^{(n)}, \mathcal{T}_2^{(n)}, \ldots, \mathcal{T}_{m_n}^{(n)}\}$,

(II) for all $i \in \{1, \ldots, m_{n+1}\}$ there exists $j \in \{1, \ldots, m_n\}$ such that $B_i^{(n+1)} \subset B_j^{(n)}$

(III) the collection $\bigcup_{n\in\mathbb{N}} \mathcal{P}^{(n)}$ generates $\mathcal{X}$.

If there exists a constant $\delta > 0$ and a sequence of **complete return words** $(w(n) \in \{1, \ldots, m_n\}^*)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \mu(\mathcal{T}_{w(n)}^{(n)}) \geq \delta$$

then $(X, \mathcal{X}, \mu, T)$ is **partially rigid**.

**Theorem** (Donoso, Mass, R. 2025) Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $(\mathcal{P}^{(n)})_{n\in\mathbb{N}}$ be a sequence of Kakutani-Roklin partition of $X$ that fulfills (I), (II) and (III).

$(X, \mathcal{X}, \mu, T)$ is **partially rigid** if and only if there exists a constant $\delta > 0$ and a sequence of **complete return words** $(w(n) \in \{1, \dots, m_n\}^*)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \mu\left(\bigcup_{u\sim w(n)} \mathcal{T}_u^{(n)}\right) \geq \delta. \quad (1)$$

Here two complete words $u = u_1 u_2 \cdots u_{\ell-1} u_\ell$, $w = w_1 w_2 \cdots w_{r-1} w_r$ are equivalent $(u \sim w)$ if

$$h_{u_1}^{(n)} + h_{u_2}^{(n)} + \cdots + h_{u_{\ell-1}}^{(n)} = h_{w_1}^{(n)} + h_{w_2}^{(n)} + \cdots + h_{w_{r-1}}^{(n)}$$

where $h_i^{(n)}$ the hight of $\mathcal{T}_i^{(n)}$.

**Remark 1:** The constant $\delta > 0$ in (1) is a partial rigidity constant for $\mu$.

**Remark 2:** The sequence of partial rigidity $(N_n)_{n\in\mathbb{N}}$ is given by $N_n = h_{u_1}^{(n)} + h_{u_2}^{(n)} + \cdots + h_{u_{|u|-1}}^{(n)}$ for some $u \sim w(n)$.

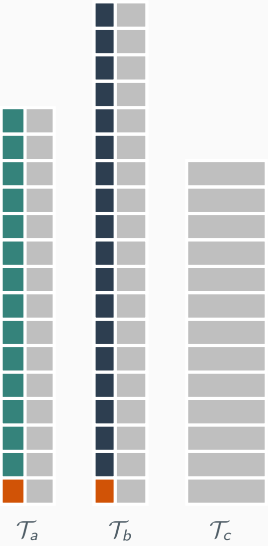
# Illustration

$w = abca$

$u = bcab$

$\mathcal{T}_w$

$\mathcal{T}_u$



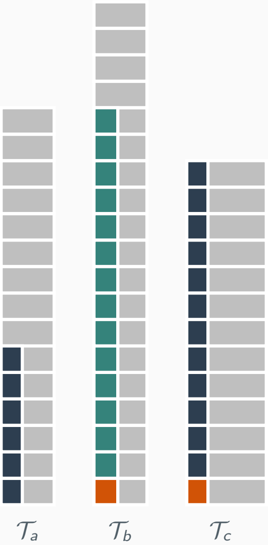
# Illustration

$w = abca$

$u = bcab$

$T^{h_a}\mathcal{T}_w$

$T^{h_b}\mathcal{T}_u$



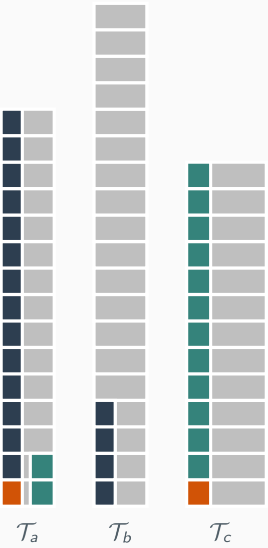
# Illustration

$w = abca$

$u = bcab$

$T^{h_a+h_b}\mathcal{T}_w$

$T^{h_b+h_c}\mathcal{T}_u$



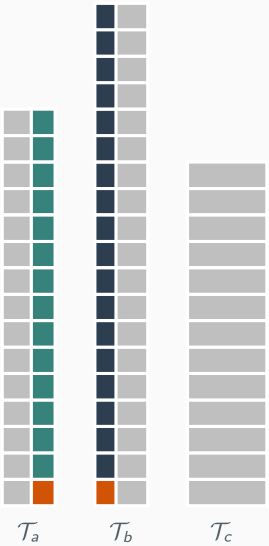
# Illustration

$w = abca$

$u = bcab$

$T^{h_a+h_b+h_c}\mathcal{T}_w$

$T^{h_b+h_c+h_a}\mathcal{T}_u$



# Applications to $\mathcal{S}$-adic subshift

- Consider $A$, a finite *alphabet*, $A^*$ the set of (finite) *words* over $A$,
- $\sigma\colon A^* \to B^*$ is a *morphism* if it is a morphism for the *concatenation* that is

$$\sigma(w_1 w_2 \cdots w_\ell) = \sigma(w_1)\sigma(w_2)\cdots\sigma(w_\ell)$$

  for $w_1, \ldots, w_\ell \in A$.
- A sequence of morphisms $\boldsymbol{\sigma} = (\sigma_n\colon A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ is called a *directive sequence*.

$$\boldsymbol{\sigma} = \left|\begin{array}{c} \vdots \\ A_3^* \\ \downarrow^{\sigma_2} \\ A_2^* \\ \downarrow^{\sigma_1} \\ A_1^* \\ \downarrow^{\sigma_0} \\ A_0^* \end{array}\right|$$

We assume $\boldsymbol{\sigma}$ will be *primitive* and *recognizable*.

$$\boldsymbol{\sigma} = \left| \begin{array}{c} \vdots \\ A_3^* \\ \downarrow^{\sigma_2} \\ A_2^* \\ \downarrow^{\sigma_1} \\ A_1^* \\ \downarrow^{\sigma_0} \\ A_0^* \end{array} \right|$$

- Fixing $n$,

$$\mathcal{L}^{(n)}(\boldsymbol{\sigma}) = \{w \in A_n^* \mid w \text{ appears in } \sigma_{[n,N)}(a) \text{ for } N > n, a \in A_N\}$$

where $\sigma_{[n,N)}(a) = \sigma_n \circ \cdots \circ \sigma_{N-1}(a)$ and

$$X_{\boldsymbol{\sigma}}^{(n)} = \{x \in A_n^{\mathbb{Z}} \mid \text{ every finite word of } x \text{ is in } \mathcal{L}^{(n)}(\boldsymbol{\sigma})\}.$$

- The *$\mathcal{S}$-adic subshift given by $\boldsymbol{\sigma}$* is $X_{\boldsymbol{\sigma}}^{(0)} = X_{\boldsymbol{\sigma}}$

**Special case:** If $\sigma \equiv \sigma \colon A^* \to A^*$ $X_\sigma$ is a *substitution subshift*,

**Example:** Thue-Morse substitution:

$$0 \mapsto 01$$
$$1 \mapsto 10$$

The elements of $X_\sigma$ locally look like,

| | |
|---:|:---|
| 0 | 1 |
| 01 | 10 |
| 0110 | 1001 |
| 01101001 | 10010110 |
| 0110100110010110 | 1001011001101001 |
| 01101001100101101001011001101001 | 10010110011010010110100110010110 |

Assume $\sigma = (\sigma_n \colon A_{n+1}^* \to A_n^*)_{n \in \mathbb{N}}$ is of finite rank, that is

$$\liminf_{n \to \infty} |A_n| < \infty$$

**Proposition** If a positive morphism $\tau$ appears infinitely many times in $\sigma$ then $X_\sigma$ is partially rigid for its unique invariant measure.

**Def.** $\tau \colon A^* \to B^*$ is positive if all letters $b \in B$ appear in $\tau(a)$ for all $a \in A$.

**Proposition** If exists $c > 0$ such that $\max_{a \in A_n} |\sigma_{[0,n)}(a)| \leq c \min_{a \in A_n} |\sigma_{[0,n)}(a)|$ for all $n \in \mathbb{N}$, then $X_\sigma$ is partially rigid for all invariant measures.

**Corollary** Finite rank **constant length** $\mathcal{S}$-adic subshift, that is

$$|\sigma_n(a)| = |\sigma_n(b)| \quad \text{for all } n \in \mathbb{N},\ a, b \in A_{n+1},$$

are partially rigid.

A morphism $\sigma\colon A^* \to B^*$ is $\ell$-consecutive if, for all $a \in A$, $\sigma(a) = b_1^{\ell_1} b_2^{\ell_2} \cdots b_k^{\ell_k}$ with $\ell_i \geq \ell$ for all $i = 1, \dots, k$.

**Proposition** If $\sigma$ is finite rank and $\sigma_n$ is $\ell$-consecutive for infinitely many $n \in \mathbb{N}$ then $X_\sigma$ is partially rigid for all invariant measures.

**Proposition** If $\sigma$ is constant-length and $\sigma_n$ is $\ell$-consecutive for infinitely many $n \in \mathbb{N}$ then $X_\sigma$ is partially rigid for all invariant measures. Moreover the constant of partial rigidity is at least $\frac{\ell-1}{\ell}$.

**Corollary** If $\sigma$ is constant-length and for all $\ell \geq 2$ there is $n \in \mathbb{N}$ such that $\sigma_n$ is $\ell$-consecutive then $X_\sigma$ is rigid for all invariant measures.

**Proposition** If $\sigma$ is constant-length and $\sigma_n$ is $\ell$-consecutive for infinitely many $n \in \mathbb{N}$ then $X_{\sigma}$ is partially rigid for all invariant measures. Moreover the constant of partial rigidity is at least $\frac{\ell-1}{\ell}$.

**Proof:** WLOG all $\sigma_n \colon A_{n+1}^* \to A_n^*$ are $\ell$-consecutive. Consider

$$\mathcal{P}^{(n)} = \{S^k \sigma_{[0,n)}([a]) : a \in A_n, 0 \le k < |\sigma_{[0,n)}(a)|\}.$$

- Take $a \in A_{n+1}$, $\sigma_n(a) = b_1^{\ell_1} b_2^{\ell_1} \cdots b_k^{\ell_k}$ consider $b \in A_n$ such that $b = b_i$ for at least one $1 \le i \le k$.
- Denote $r = \#\{i \mid b = b_i\}$. By $\ell$-consecutive, $\ell_i \ge \ell$ and

$$\frac{|\sigma_n(a)|_b - |\sigma_n(a)|_{bb}}{|\sigma_n(a)|_b} = \frac{r}{\sum_{i:\ b=b_i} \ell_i} \le \frac{r}{r\ell} = \frac{1}{\ell}.$$

where $|w|_u = \# u$ appears in $w$. Thus

$$\mu(\mathcal{T}_{bb}^{(n)} \cap \mathcal{T}_a^{(n+1)}) \ge \frac{|\sigma_n(a)|_{bb}}{|\sigma_n(a)|_b} \mu(\mathcal{T}_b^{(n)} \cap \mathcal{T}_a^{(n+1)}) \ge \frac{\ell-1}{\ell} \mu(\mathcal{T}_b^{(n)} \cap \mathcal{T}_a^{(n+1)}).$$

**Proposition** If $\sigma$ is constant-length and $\sigma_n$ is $\ell$-consecutive for infinitely many $n \in \mathbb{N}$ then $X_\sigma$ is partially rigid for all invariant measures. Moreover the constant of partial rigidity is at least $\frac{\ell-1}{\ell}$.

**Proof:**

- Thus summing over $A_{n+1}$

$$\mu(\mathcal{T}_{bb}^{(n)}) = \sum_{a \in A_{n+1}} \mu(\mathcal{T}_{bb}^{(n)} \cap \mathcal{T}_a^{(n+1)}) \geq \frac{\ell - 1}{\ell} \sum_{a \in A_{n+1}} \mu(\mathcal{T}_b^{(n)} \cap \mathcal{T}_a^{(n+1)}) = \frac{\ell - 1}{\ell} \mu(\mathcal{T}_b^{(n)}).$$

- By constant length constant lengths, for every $b_1, b_2 \in A_n$, $b_1 b_1 \sim b_2 b_2$.

$$\mu\left(\bigcup_{b \in A_n} \mathcal{T}_{bb}^{(n)}\right) = \sum_{b \in A_n} \mu(\mathcal{T}_{bb}^{(n)}) \geq \frac{\ell - 1}{\ell} \sum_{b \in A_n} \mu(\mathcal{T}_b^{(n)}) = \frac{\ell - 1}{\ell}$$

**Proposition** Let $(X, S)$ be a **linearly recurrent** subshift and let $\mu$ be its unique invariant measure. If $(X, \mathcal{B}, \mu, S)$ is **rigid**, then

$$\limsup_{n\to\infty} \frac{q(n)}{p(n)} = 1 \tag{2}$$

where

- $p(n) = \#$ words of length $n$ in $X$,
- $q(n) = \#$ complete return words of length $n$ in $X$.

**Proposition** If $(X, S)$ is a **constant length substitution** subshift, then rigidity is equivalent to (2).

# Partial rigidity rate

**Definition**

$$\delta_\mu(T) = \sup\{\delta > 0 \mid \delta \text{ is a constant of partial rigidity}\}$$

is the *partial rigidity rate*.

Sometime we just write $\delta_\mu$.

**Proposition** For $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$

- If $X$ and $Y$ are isomorphic then $\delta_\mu(T) = \delta_\nu(S)$
- If $\pi : X \to Y$ a factor $\delta_\mu(T) \leq \delta_\nu(S)$
- $\delta_{\mu\times\mu}(T \times T) = (\delta_\mu(T))^2$

**Recall:**

**Theorem** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $(\mathcal{P}^{(n)})_{n\in\mathbb{N}}$ be a sequence of Kakutani-Roklin partition of $X$ that fulfills (I), (II) and (III).

$(X, \mathcal{X}, \mu, T)$ is **partially rigid** if and only if there exists a constant $\delta > 0$ and a sequence of **complete return words** $(w(n) \in \{1, \dots, m_n\}^*)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \mu\left(\bigcup_{u\sim w(n)} \mathcal{T}_u^{(n)}\right) \geq \delta.$$

**Now:**

**Theorem** Under the same hypothesis, there exists a sequence of **complete return words** $(w(n) \in \{1, \dots, m_n\}^*)_{n\in\mathbb{N}}$ such that

$$\delta_\mu(T) = \lim_{n\to\infty} \mu\left(\bigcup_{u\sim w(n)} \mathcal{T}_u^{(n)}\right) \tag{3}$$

# Special case, substitution subshift

**Corollary** Let $\sigma : A^* \to A^*$ be a constant-length substitution and $\mu$ the unique invariant measure of $X_\sigma$.

$$\delta_\mu = \sup_{\ell \geq 2} \left\{ \sum_{w:\ |w|=\ell,\ w_1 = w_\ell} \mu(w) \right\}.$$

**Proposition** For the Thue-Morse substitution subshift:

$$\delta_\mu = \frac{2}{3}$$

with partial rigidity sequence $(3 \cdot 2^n)_{n \in \mathbb{N}}$.

**Proposition** For $L \geq 6$, let $\zeta_L \colon \{a, b\}^* \to \{a, b\}^*$ be the substitution given by

$$\zeta_L(a) = ab^{L-1}$$
$$\zeta_L(b) = ba^{L-1},$$

then for the invariant measure of the substitution subshift $X_{\zeta_L}$

$$\delta_\mu = \frac{L-1}{L+1}$$

with partial rigidity sequence $(L^n)_{n\in\mathbb{N}}$.

**Corollary** For every $\delta \in [0, 1]$ there is a measure preserving system $(X, \mathcal{X}, \mu, T)$ such that $\delta = \delta_\mu(T)$.

# **System with** $0 < \delta_{\mu_0} < \ldots < \delta_{\mu_{d-1}} < 1$

# Partial rigidity rates

**Proposition** Let $\sigma = (\sigma_n\colon A_{n+1}^* \to A_n^*)_{n\in\mathbb{N}}$ be a constant length directive sequence. Let $\mu$ be an ergodic measure on $X_\sigma$. Then

$$\delta_{\mu^{(n)}} = \delta_\mu \quad \text{for all } n \in \mathbb{N}$$

where $\mu^{(n)}$ is the corresponding invariant measure in $X_\sigma^{(n)}$.

**Idea:** for a constant length $\mathcal{S}$-adic subshift and $\mu$ ergodic measure,

- if we know $\delta_{\mu^{(5)}}$ we know $\delta_\mu$,
- if we know $\delta_{\mu^{(100)}}$ we know $\delta_\mu$,
- if we know $\delta_{\mu^{(10000)}}$ we know $\delta_\mu$,
- if we know $\delta_{\mu^{(\infty)}}$???? we know $\delta_\mu$.

**Construction** Take the alphabet $A = \{a, b, 0, 1\}$. Let $\kappa \colon A^* \to A^*$ be the function (**not morphism**) such that $u \in A^*$,

$$\kappa(ua) = u0 \quad \kappa(ub) = u1 \quad \kappa(u0) = ua \quad \text{and} \quad \kappa(u1) = ub$$

Consider $\zeta_6 \colon \{a, b\}^* \to \{a, b\}^*$ and $\zeta_{36} \colon \{0, 1\}^* \to \{0, 1\}^*$

$$\begin{aligned} \zeta_6(a) &= aaaaab & \zeta_6(b) &= bbbbba \\ \zeta_{36}(0) &= 0^{35}1 & \zeta_{36}(1) &= 1^{35}0. \end{aligned}$$

Define $\sigma_n \colon A^* \to A^*$ be given by

$$\begin{aligned} \sigma_n(a) &= \kappa(\zeta_6^{2n}(a)) \\ \sigma_n(b) &= \kappa(\zeta_6^{2n}(b)) \\ \sigma_n(0) &= \kappa(\zeta_{36}^{n}(0)) \\ \sigma_n(1) &= \kappa(\zeta_{36}^{n}(1)). \end{aligned}$$

**Remark** $\sigma_n$ is of constant-length with $|\sigma_n| = 36^n$.

**The idea is to "glue" systems for which we know the partial rigidity rate.**

# Partial rigidity rates

Take $\sigma_n \colon A^* \to A^*$ with

$$\begin{aligned}\sigma_n(a) &= \kappa(\zeta_6^{2n}(a))\\ \sigma_n(b) &= \kappa(\zeta_6^{2n}(b))\\ \sigma_n(0) &= \kappa(\zeta_{36}^{n}(0))\\ \sigma_n(1) &= \kappa(\zeta_{36}^{n}(1)).\end{aligned}$$

**Theorem** (R. 2025) For $\boldsymbol{\sigma} = (\sigma_n \colon A^* \to A^*)_{n\in\mathbb{N}}$, then $(X_{\boldsymbol{\sigma}}, S)$ has two ergodic measure $\{\mu_1, \mu_2\}$ such that

$$\begin{aligned}\delta_{\mu_1} &= \frac{6-1}{6+1}\\ \delta_{\mu_2} &= \frac{36-1}{36+1}\end{aligned}$$

Moreover, the partial rigidity sequence associated to $\delta_{\mu_1}$ and $\delta_{\mu_2}$ are equal.

This strategy can be done for $d \geq 2$ measures using $\{a_1, b_1, a_2, b_2, \ldots, a_d, b_d\}$ as alphabet.

# Open problems

## Open problems

- Is the condition $\limsup_{n\to\infty} \frac{q(n)}{p(n)} = 1$ necessary/equivalent to rigidity for a larger family of systems?
- Rigidity for Pisot substitution (ask in Bruin, Karpel, Oprocha, Radinger 2025)
- For every zero entropy Cantor system find an orbit equivalent system that has at least one partially rigid measure (ask in BKOR 2025). Already known for constant-length $\mathcal{S}$-adic subshift and for finite topological rank.
- Characterize or compute the partial rigidity rate of some systems of interest. Similarly construct an algorithm to compute it for some special cases.
- Is $\Delta := \{\delta_\mu \mid (X, \mathcal{X}, \mu, T) \text{ is ergodic } \} = [0,1]$?
- Is $\Delta_s := \{\delta_\mu \mid \text{for substitutions } \}$ dense in $[0,1]$?
- Construct a minimal system $(X, T)$ with distinct partial rigidity rates $\delta_{\mu_1} < \ldots < \delta_{\mu_d}$ that is weakly-mixing for all measures.
- Find a uniquely ergodic system $(X, T)$ with $\delta_\mu(T) \notin \mathbb{Q}$.

# Merci!