Random Pisot substitutions and their Rauzy fractals

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Substitutions

- $\mathcal{A}$ — finite alphabet
- $\theta\colon \mathcal{A} \to \mathcal{A}^+$ — substitution
- All substitutions have a *fixed point* — $f(w) = w$
- $(M_\theta)_{ij} \coloneqq |\theta(a_j)|_{a_i}$ — substitution matrix

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Example:

- $\tau\colon a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$
- $a \mapsto ab \mapsto abac \mapsto abacaba \mapsto \cdots$
- $w = abacabaabacababacabaabacabacabaab\cdots$
- $M_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

The Substitution Matrix

If there is $k \geq 1$ such that $M_\theta^k > 0$, we call θ *primitive*

Perron–Frobenius $\implies$ exists unique simple real eigenvalue $\lambda_{PF} > 1$ with eigenvector $\mathbf{R} > 0$, also $\lambda_{PF} > |\lambda'|$

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Tribonacci:

$$M_\tau^3 = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \lambda = \lambda_{PF} \simeq 1.839 \quad \lambda^3 - \lambda^2 - \lambda - 1 = 0$$

$$\mathbf{R} = (\lambda^{-1}, \lambda^{-2}, \lambda^{-3})^T \simeq (0.544, 0.296, 0.161)^T$$

Pisot Substitutions

We say that a substitution is *Pisot* if λ_{PF} is a Pisot number, and *irreducible* if $\deg \lambda_{PF} = \#\mathcal{A}$

Fact: A substitution is irreducible Pisot if and only if for all $\lambda' \neq \lambda_{PF}$

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All the non-PF eigenvectors span a codimension-1 subspace H, which we call the *contracting hyperplane*

We construct a geometric object on H called the *Rauzy fractal*

C-balancedness and Rauzy fractals

Let w be an infinite word with well-defined letter-frequencies $\mathbf{R}_i$

Then w is *C-balanced* (on letters) or has *bounded discrepancy* if there is a $C > 0$ such that for all subwords u,

$$\big||u|_{a_i} - \mathbf{R}_i|u|\big| \leq C$$

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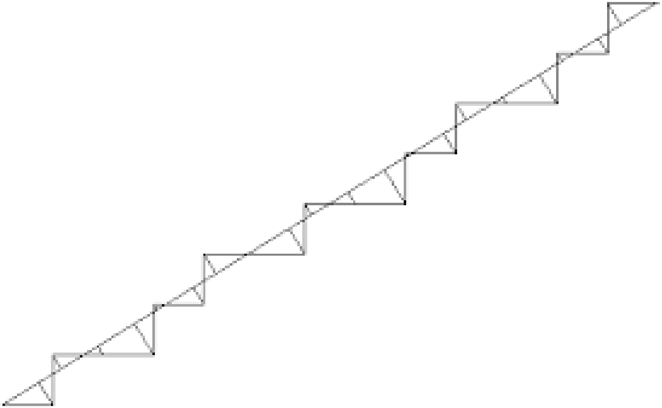
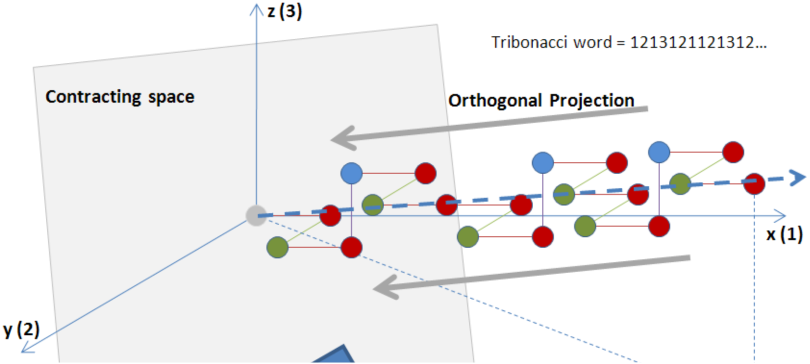
Notation: $[u] = (|u|_{a_1}, \ldots, |u|_{a_d})$ is the abelianisation of the word u

> If w is C-balanced, then the lattice vectors
>
> $$[w_1], [w_1 w_2], [w_1 w_2 w_3], \ldots$$
>
> all remain within a bounded neighbourhood of the ray spanned by the frequency vector $\mathbf{R}$

Fibonacci staircase



Tribonacci staircase



[*Shamelessly stolen from wikipedia*]

Fact

All irreducible Pisot substitutions are C-balanced

Define

$$\mathcal{R}(\theta) = \mathcal{R}(w) := \overline{\{\mathrm{proj}_H(w_{[0,n]}) \mid n \geq 0\}},$$

the *Rauzy fractal* of w.

$$\tau : \begin{cases} a \mapsto ab \\ b \mapsto ac \\ c \mapsto a \end{cases} \qquad \tilde{\tau} : \begin{cases} a \mapsto {\color{red} ba} \\ b \mapsto ac \\ c \mapsto a \end{cases}$$

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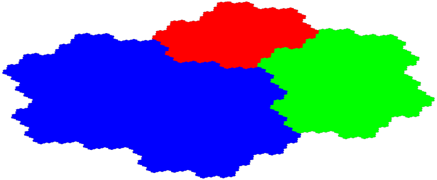

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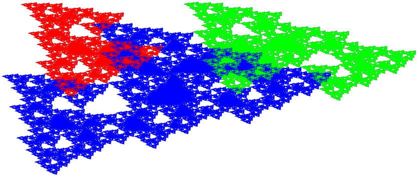

Twisted Tribonaci

Rauzy fractals

$\mathcal{R}(\theta)$ is a compact subset of $\mathbb{R}^{d-1}$ equal to closure of its interior. Always contains an open ball. Not always simply connected or even connected.

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Tribonacci:

$$\tau\colon a \mapsto ab,\ b \mapsto ac,\ c \mapsto a$$

$$\mathcal{R}(\tau) = \mathcal{R}_a \cup \mathcal{R}_b \cup \mathcal{R}_c$$

$$\mathcal{R}_a = h(\mathcal{R}_a) \cup h(\mathcal{R}_b) \cup h(\mathcal{R}_c), \quad \mathcal{R}_b = h(\mathcal{R}_a) + \mathbf{v}_a, \quad \mathcal{R}_c = h(\mathcal{R}_b) + \mathbf{v}_a$$

$h := \mathrm{proj}_H(M)$, $\mathbf{v}_a := \mathrm{proj}_H(e_a)$, $\mathbf{v}_b := \mathrm{proj}_H(e_b)$

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Twisted tribonacci:

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The Pisot Conjecture

The *Pisot conjecture* says that for all irreducible Pisot θ, the Rauzy fractal $\mathcal{R}(\theta)$ tiles the plane with translation group

$$\mathcal{L} = \{n(\mathbf{v}_a - \mathbf{v}_b) + m(\mathbf{v}_a - \mathbf{v}_c) \mid (n, m) \in \mathbb{Z}^2\}$$

So $\bigcup_{\mathbf{v} \in \mathcal{L}} \mathcal{R}(\theta) + \mathbf{v}$ is a tiling of H

Equivalently, the orbit closure subshift $X_\theta := \overline{\{\sigma^n(w) \mid n \in \mathbb{N}\}}$ factors onto the torus $H/\mathcal{L}$ almost everywhere one-to-one

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It is known that $\bigcup_{\mathbf{v}\in\mathcal{L}} \mathcal{R}(\theta) + \mathbf{v}$ is a k-fold multitiling of H so the difficulty is in showing that $k = 1$

The conjecture was solved for $\mathcal{A} = \{a, b\}$ by [Barge–Diamond '02] and [Hollander–Solomyak '03]

Barge substitutions

In 2016, Barge proved an important case of the conjecture, including for all *β-substitutions* (substitutions that code IETs of the form $x \mapsto \beta x - \lfloor \beta x \rfloor$)

A *Barge substitution* is a substitution θ of the form

$$\theta\colon \begin{cases} a & \mapsto & x \cdots \rho(a) \\ b & \mapsto & x \cdots \rho(b) \\ c & \mapsto & x \cdots \rho(c) \end{cases}$$

where $x \in \mathcal{A}$ is a fixed letter and ρ is permutation on $\mathcal{A}$

Barge proved that every irreducible Pisot Barge substitution satisfies the Pisot conjecture

Barge substitutions

Observation: For any primitive substitution θ, there exists a *Barge cousin* $\hat{\theta}$, which is Barge, and for which $M_\theta^k = M_{\hat{\theta}}$ for some $k \geq 1$

Ex: Tribonacci is a Barge cousin of Twisted Tribonacci

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Our Motivation

Maybe there's a way of proving the Pisot conjecture for θ by using the fact that $\hat{\theta}$ satisfies the conjecture and then 'transferring' the tiling property across...

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Spoiler: We haven't proven the Pisot conjecture...

Random substitutions

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Locally mix tribonacci with twisted tribonacci (Random tribonacci)

$$\tau_{\mathbf{p}}: \begin{cases} a & \mapsto & \{ab, ba\} \qquad \text{with probabilities } (p, 1-p) \\ b & \mapsto & \{ac\} \\ c & \mapsto & \{a\} \end{cases}$$

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Choices are independent for each letter:

$$a \mapsto ab \mapsto \overbrace{ba}^{\tau_{\mathbf{p}}(a)} ac \mapsto ac \overbrace{ab}^{\tau_{\mathbf{p}}(a)} \overbrace{ba}^{\tau_{\mathbf{p}}(a)} a \mapsto baabaacacabba \mapsto \cdots$$

$$\mathcal{L}_{\tau_{\mathbf{p}}} := \{u \mid u \triangleleft v \in \tau_{\mathbf{p}}^k(a),\ k \geq 0,\ a \in \mathcal{A}\}$$

$$X_{\tau_{\mathbf{p}}} := \{x \mid u \triangleleft x \implies u \in \mathcal{L}_{\tau_{\mathbf{p}}}\}$$

Basic properties

Thm. [R.–Spindeler '20]

The following properties hold for $X_{\vartheta_{\mathbf{p}}}$ for primitive $\vartheta_{\mathbf{p}}$:

- Cantor set
- Either no periodic points or periodic points are dense
- Uncountably many minimal components
- Uncountably many ergodic prob. measure
- Canonical measure $\mu_{\mathbf{p}}$ induced by probabilities $\mathbf{p}$ (shown to be ergodic [Gohlke–Spindeler '20])
- Almost all orbits are dense
- Positive entropy

Further entropy results: topological entropy [Gohlke '20], [Mitchell '23]; measure theoretic entropy [Gohlke–Mitchell–R.–Samuel '23]

A random substitution is *compatible* if

$$u, v \in \vartheta_{\mathbf{p}}(a) \implies [u] = [v]$$

Ensures we have well-defined **constant** substitution matrix $M_{\vartheta_{\mathbf{p}}}$

Allows us to 'locally mix' two substitutions $\theta, \hat{\theta}$ for which $M_\theta = M_{\hat{\theta}}$

Basic properties

Let $\vartheta_{\mathbf{p}}$ be a primitive, compatible random substitution
Write $\lambda_{PF} > |\lambda_2| \geq \cdots \geq |\lambda_d|$

Thm. [Miro–R.–Sadun–Tadeo '20]

- If $|\lambda_2| < 1$, then $X_{\vartheta_{\mathbf{p}}}$ is C-balanced.
- If $|\lambda_2| > 1$, then $X_{\vartheta_{\mathbf{p}}}$ is not C-balanced.
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Open: Classify C-balancedness when $|\lambda_2| = 1$.
In deterministic setting, this is solved, but quite complicated [Adamczewski '03]

Therefore Rauzy fractals exist for irreducible Pisot random substitutions and it turns out they are also attractors of a GIFS

Random tribonacci:

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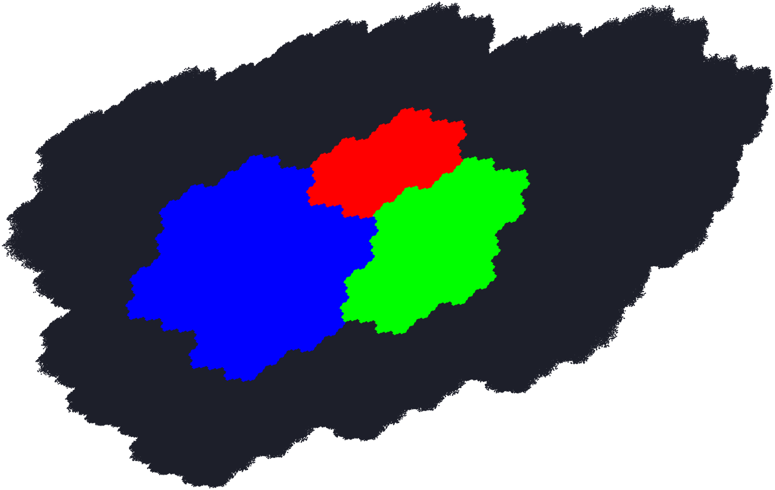
trib.

twisted trib.

both

$\tau_{\mathbf{p}}\colon a \mapsto \{ab, ba\},\ b \mapsto \{ac\},\ c \mapsto \{a\}$



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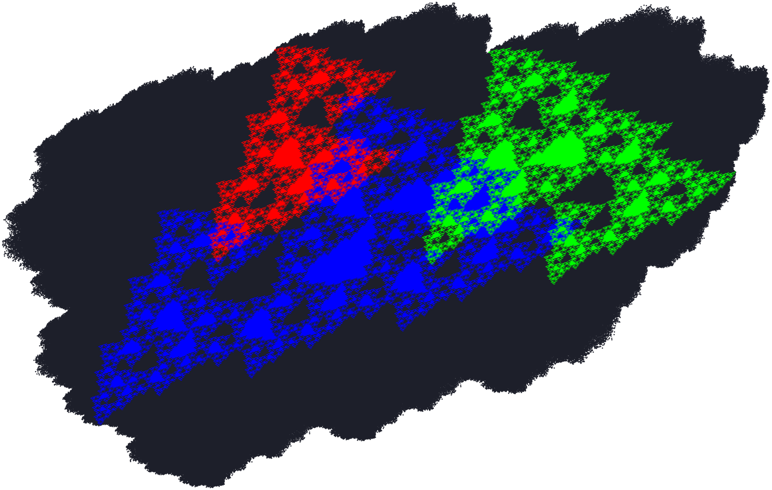
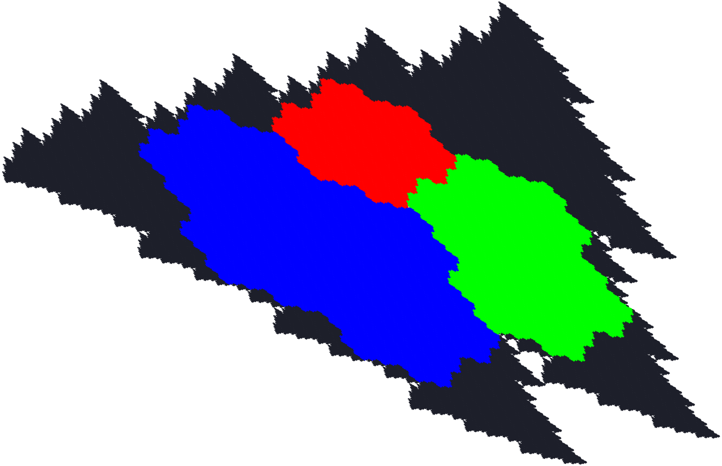


$\tilde{\tau}_{\mathbf{p}} \colon a \mapsto \{ab\},\ b \mapsto \{ac, ca\},\ c \mapsto \{a\}$



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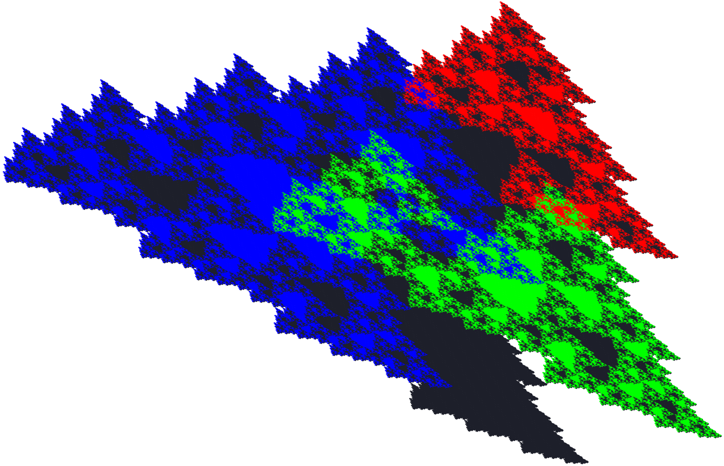

The Mother Substitution

$\mathcal{R}(\vartheta_{\mathbf{p}})$ is independent of the probabilities $\mathbf{p}$ (not surprising, almost every element of X_ϑ is dense w.r.t. measure induced by $\mathbf{p}$)

In fact, $\mathcal{R}(\vartheta_{\mathbf{p}})$ is the Rauzy fractal for a deterministic substitution, but on a larger alphabet (and necessarily reducible)

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Ex: For the random Fibonacci substitution $\vartheta_{\mathbf{p}} \colon a \mapsto \{ab, ba\},\ b \mapsto \{a\}$, the mother substitution is

$$A \mapsto AB,\ \tilde{A} \mapsto B\tilde{A},$$
$$B \mapsto C,\ C \mapsto DEF,$$
$$D \mapsto C,\ E \mapsto B,\ F \mapsto C$$

Rauzy Measures for Random Substitutions

As $\mathcal{R}(\vartheta_{\mathbf{p}})$ is independent of the probabilities $\mathbf{p}$, suggests we're not looking at the right object

Instead, let's weight lattice points by taking the *expected staircase*
Now take a limit of normalised projections of longer and longer segments
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Thm. [GMRS '24] For a.e. $w \in X_\vartheta$, the Rauzy measure is equal to

$$\tilde{\nu}_{\vartheta_{\mathbf{p}}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\mathrm{proj}_H(w_{[0,i]})}$$

where the limit is taken in the weak topology

What this means is that we can experimentally generate long words w and visualise the density of the measure

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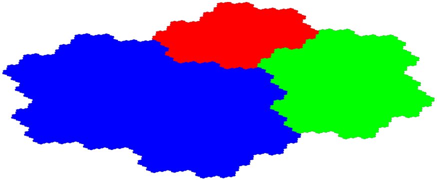
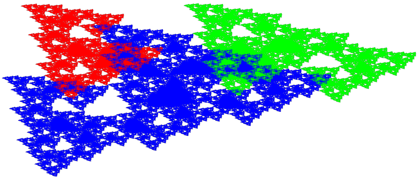


Tribonacci



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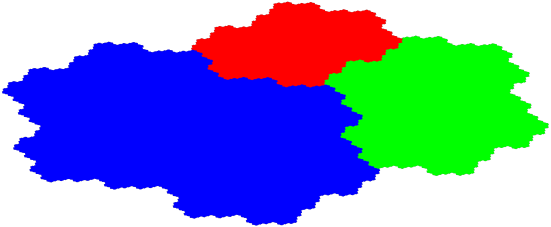

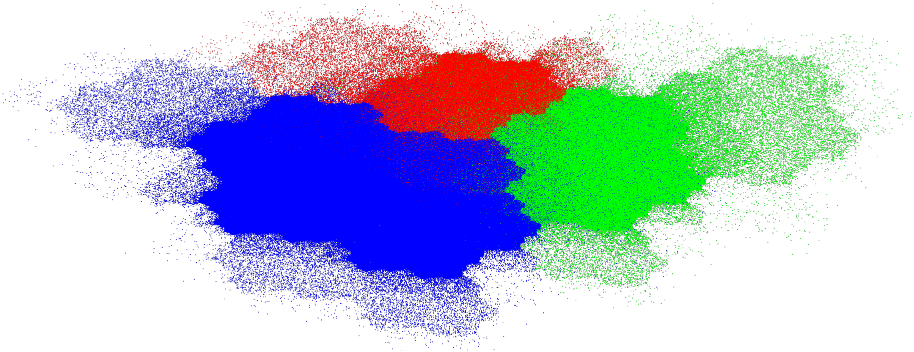

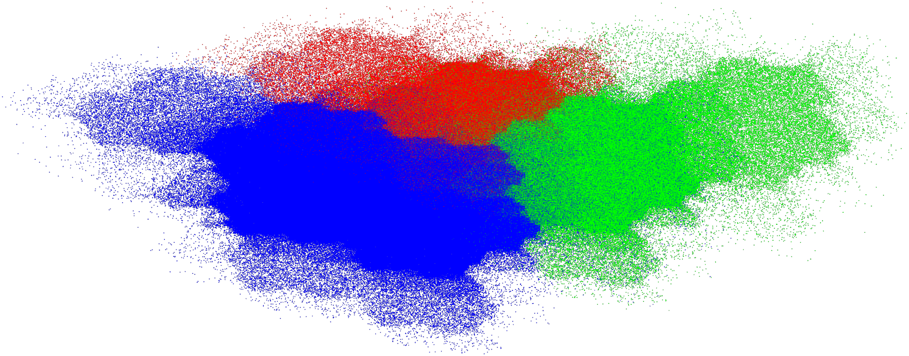

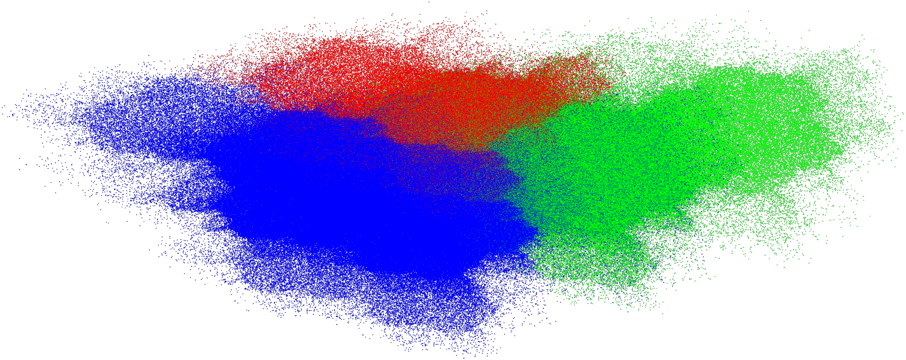

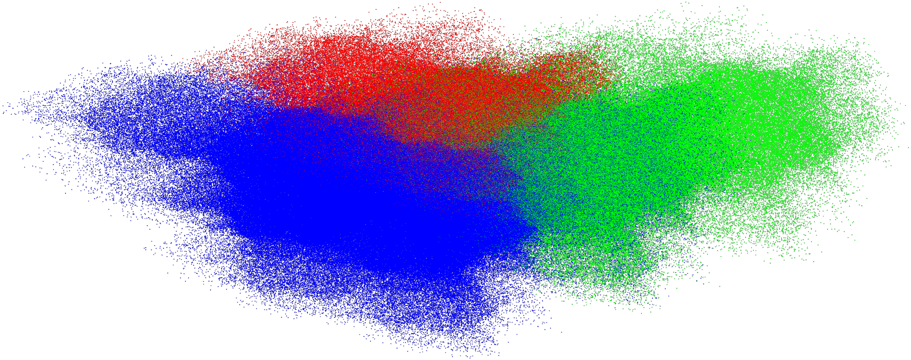

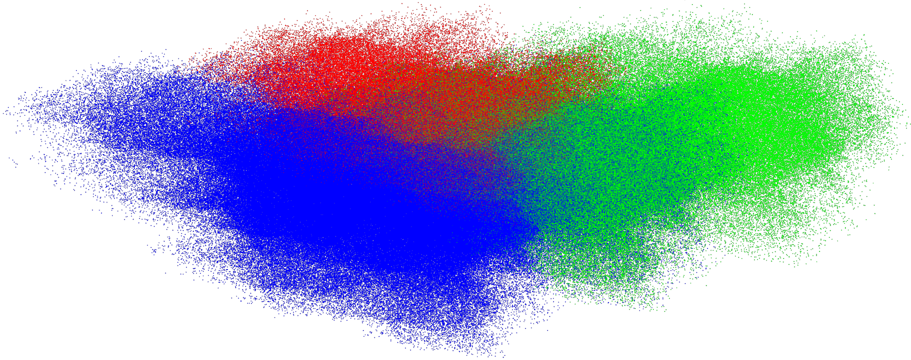

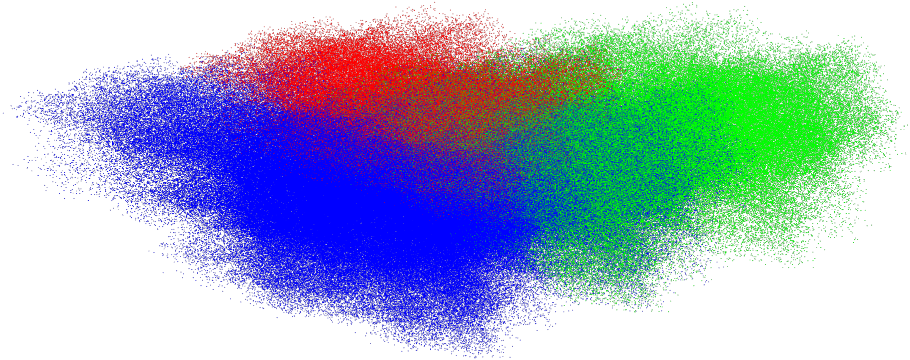

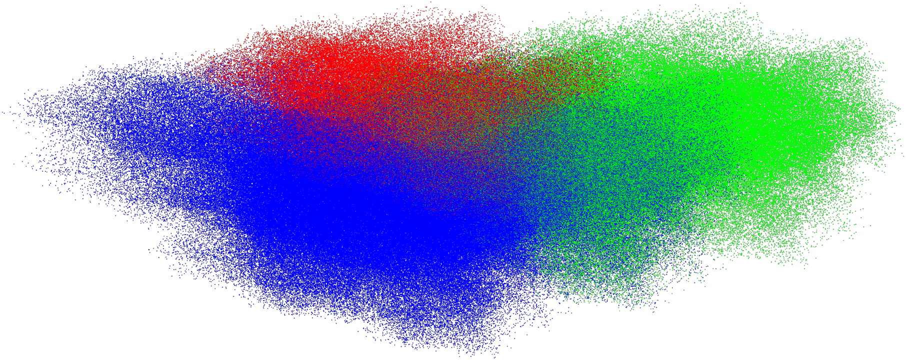

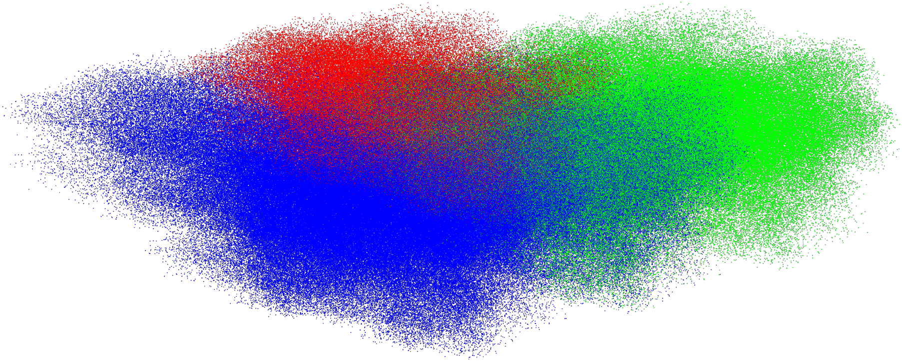

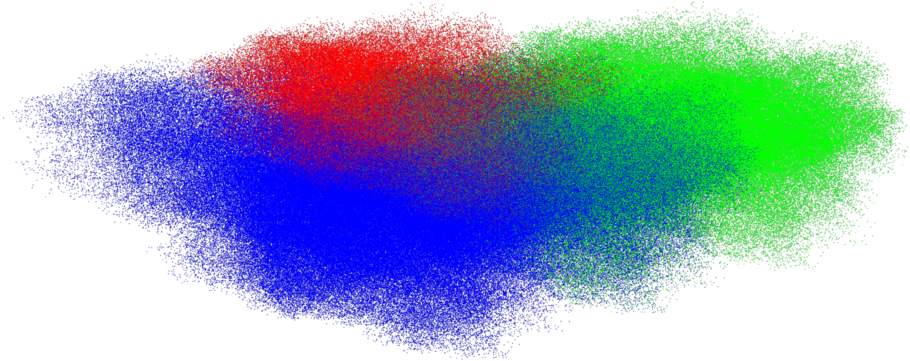

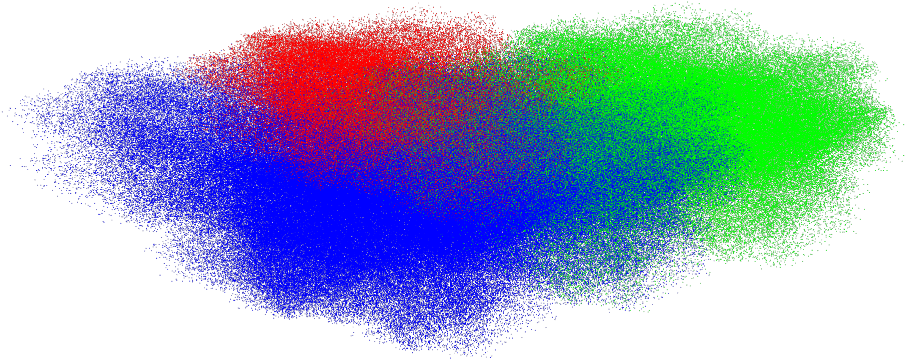

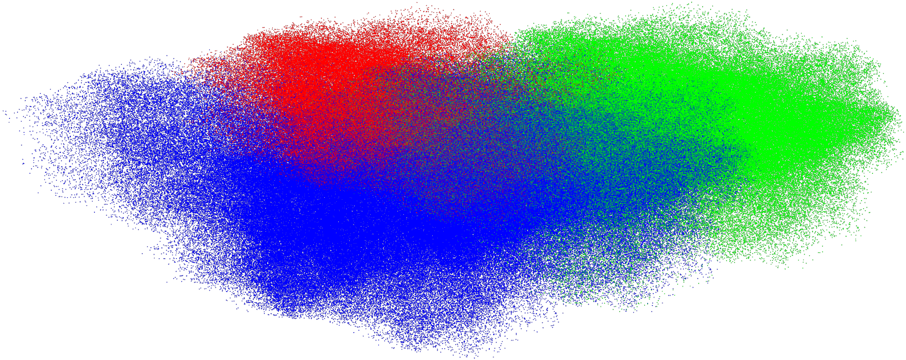

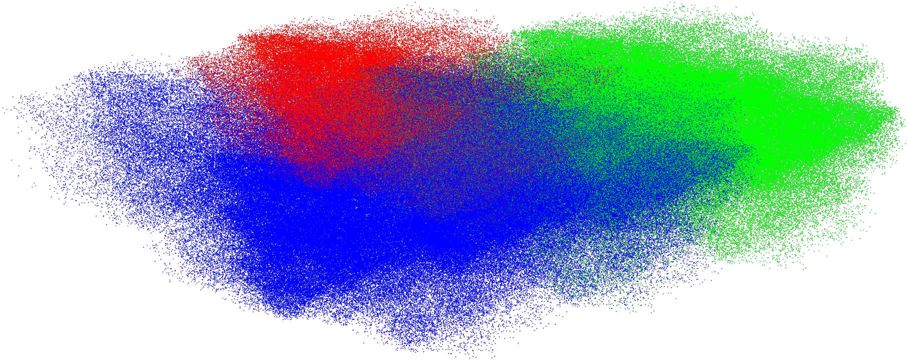

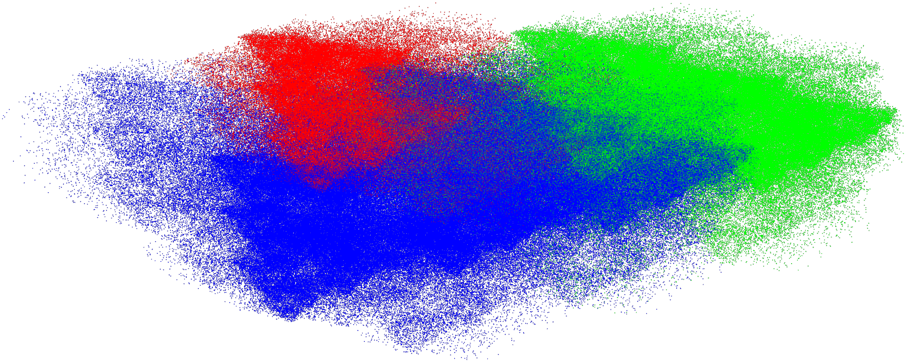

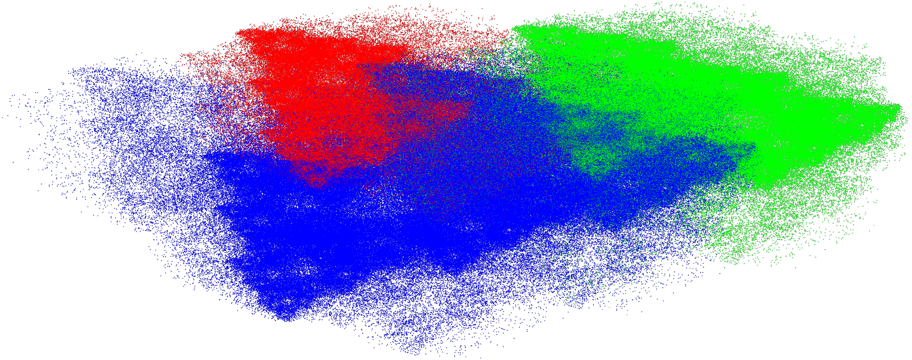

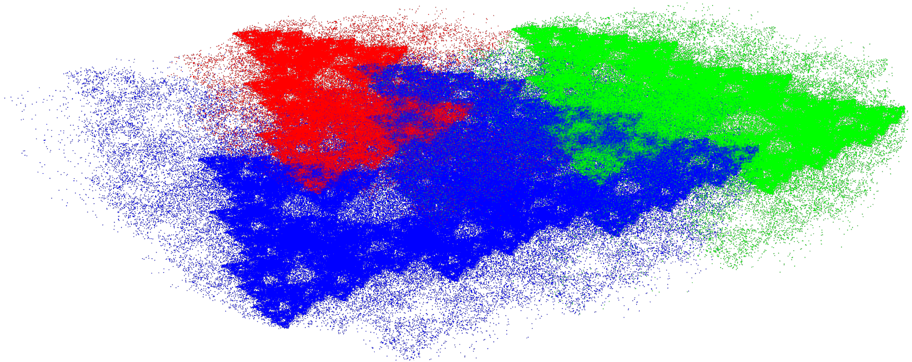

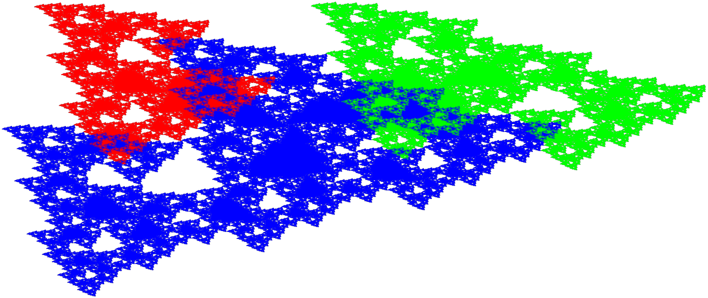

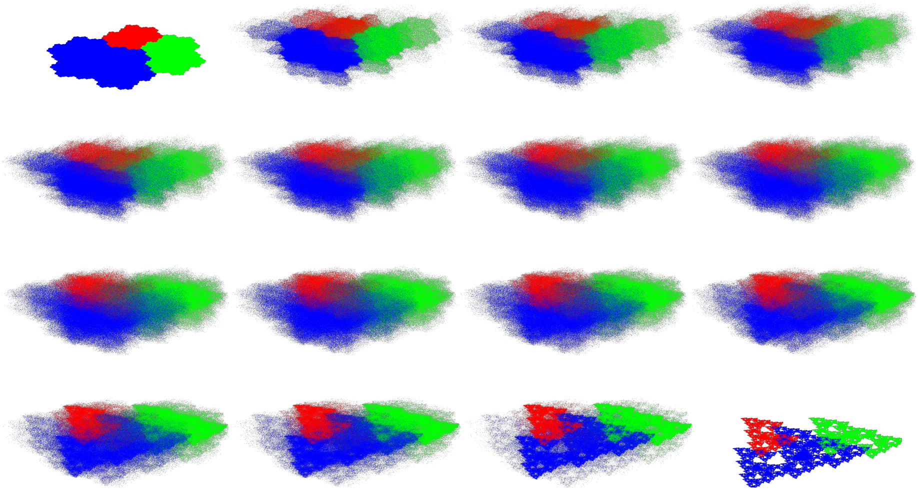

- Topologically, $\mathcal{R}(\vartheta_{\mathbf{p}})$ does not depend on $p \in (0,1)$
- Difference in images therefore indicates different densities of Rauzy measures $\nu_{\vartheta_{\mathbf{p}}}$ as p changes

Thm. [GMRS '24] The function $\mathbf{p} \mapsto \nu_{\vartheta_{\mathbf{p}}}$ is continuous with respect to the Monge—Kantorovich metric

Thm. [GMRS '24] The Rauzy measure is the unique attractor of a natural weighted GIFS that is analogous to the GIFS for generating Rauzy fractals but with edges weights corresponding to generating probabilities

Short S-adic detour

- $S = (\theta_0, \ldots, \theta_{k-1})$ — irreducible Pisot substitutions with same matrix
- $\vartheta_{\mathbf{p}}$ — local mixture of S as random substitution
- $X_{S,d}$ — S-adic system
- $\nu_{S,d}$ — associated Rauzy measure $(= \mathrm{Leb}|_{\mathcal{R}_{S,d}})$
- $\rho_{\mathbf{p}}$ — Bernouilli measure on $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$

Thm. [GMRS '24] The expectation of the Rauzy measures $\nu_{S,d}$ w.r.t $\rho_{\mathbf{p}}$ is the Rauzy measure $\nu_{\vartheta_{\mathbf{p}}}$

$$\mathbb{E}_{\rho_{\mathbf{p}}}[\nu_{S,d}] = \nu_{\vartheta_{\mathbf{p}}}$$

Different Perspectives

Summary of all the different ways to view $\nu_{\vartheta_{\mathbf{p}}}$:

- Projected 'average staircase'
- Normalised projected staircase for generic point
- Attractor of a weighted GIFS
- Average of S-adic Rauzy fractals
- (not mentioned here) pullback of factor map to MEGF

Tempting to frame in the context of the Pisot conjecture

- Naive approach:
  - Construct $\hat{\theta}$, the Barge cousin of θ
  - Construct random substitution ϑ — local mixture of θ^k and $\hat{\theta}$
  - We know that $\mathcal{R}(\hat{\theta})$ tiles the plane [Barge, '16]
  - We also know that $\nu_\theta = \mathrm{Leb}|_{\mathcal{R}(\theta)}$ and $\nu_{\hat{\theta}} = \mathrm{Leb}|_{\mathcal{R}(\hat{\theta})}$ are uniformly distributed on the respective Rauzy fractals
  - Show tilability of $\nu_{\vartheta_\mathbf{p}}$ is invariant as p ranges smoothly from 1 to 0
  - Conclude that $\mathcal{R}(\theta^k) = \mathcal{R}(\theta)$ tiles the plane

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That's exactly what we get

Thm. [GMRS '24] Let ϑ be a Pisot random substitution

$$\sum_{\mathbf{v} \in \mathcal{L}} \nu_{\vartheta_{\mathbf{p}}} + \mathbf{v} = D \,\mathrm{Leb},$$

where D is a uniform constant independent of $\mathbf{p}$

As a corollary, this implies that $\nu_{\vartheta_{\mathbf{p}}}$ is absolutely continuous w.r.t. Lebesgue

The problem is that even if the Rauzy measure tiles Leb, we don't know the supports of the individual measures

We need better control on the supports of the Rauzy measures

Unfortunately, the supports are not well behaved — three regimes

$$p = 0, \quad 0 < p < 1, \quad p = 1,$$

and the jump from one to another is strictly discontinuous because $\operatorname{supp} \nu_\theta \subsetneq \operatorname{supp} \nu_{\vartheta_{\mathbf{p}}}$

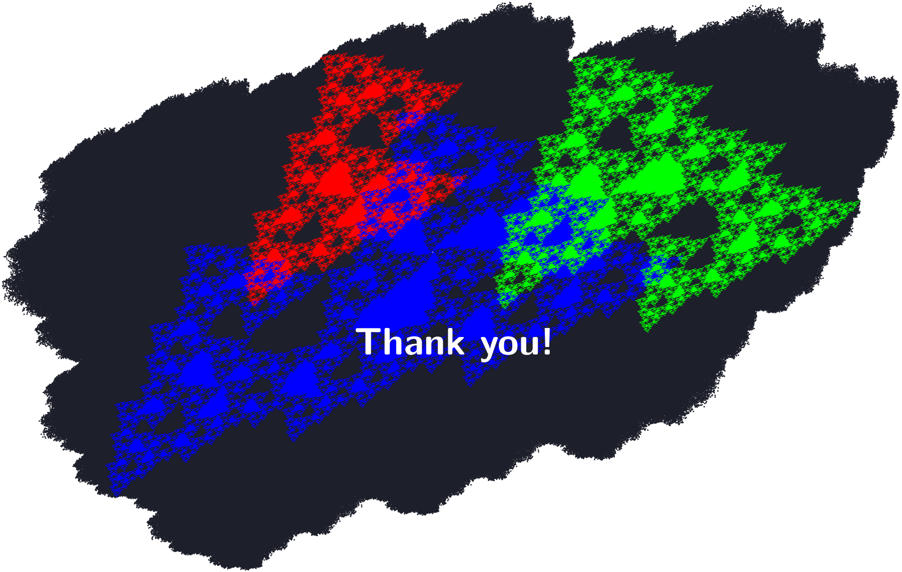
But we can say something!

As $\mathrm{Leb}(\mathcal{R}(\theta)) \in \mathbb{N}$ and $\mathrm{Leb}(\mathcal{R}(\theta)) \leq \mathrm{Leb}(\mathcal{R}(\vartheta))$, then we have the following:

Thm. [GMRS '24] Let ϑ be a Pisot random substitution such that $\mathrm{Leb}(\mathcal{R}(\vartheta)) < 2$. Then for all marginals θ (in fact any S-adic) of ϑ, we have $\mathrm{Leb}(\mathcal{R}(\theta)) = 1$ and so θ satisfies the Pisot conjecture.

Condition doesn't always hold, but examples exist

Ex: $\vartheta\colon a \mapsto \{aab\}, b \mapsto \{ab, ba\}$ $\qquad \mathcal{R}(\vartheta) = [-\tau^{-1}, 1]$


Thank you!